Formative Conversation Starters (Preview)

Student understanding is more about growing than it is about getting. As educators, we might speak of some students "getting it" and other students not. This conclusion, though, is not fair to students. A student who may not seem to "get it" does understand *something*, even if that *something* may not yet have grown into a robust web of thinking. It is important to try to get at how students are thinking as we consider how to best meet learning needs. In these activities, we probe how students might be thinking about big ideas that transcend grade-levels and standards. Recent efforts in education have focused on "unpacking" standards. Here, though, we encourage users to re-pack the mathematics in a holistic way.

Purpose: The purpose of Formative Conversation Starters is to help teachers reveal student understanding about key ideas in mathematics and to better understand their students' ways of thinking.

Audience: The intended audience is teachers, who will use these questioning strategies with students.

Application: Teachers may wish to use these conversation starters in one-on-one conferences with individual students, small groups, or with a whole class.

The goal of these activities is not to tell students what to do, nor is it about trying to immediately correct misconceptions. Instead, the challenge here is to *listen, as* the Conversation Starters provide opportunities for students to communicate how they are thinking about mathematics. A way of thinking is bigger than a single question, so we take the time here to ask about ideas from multiple directions. From this, teachers might better understand how students are thinking, and students themselves might recognize their own points of dissonance.

BINS (Big Ideas to Nurture Sense-making)

As you read through the sample clusters, you will notice that we draw attention to a few specific mathematical ideas. These BINS (Big Ideas to Nurture Sense-making) correspond to important ways of thinking that all students should develop and continue to refine from year-to-year. They transcend a single standard or a single year. They include:

- **<u>Operations:</u>** Students begin to develop meanings for operations in kindergarten (e.g., addition is putting together). As they progress, the numbers involved—and operational meanings—extend. Students should develop ways of thinking that enable them to connect operation meanings to everyday use of those operations. Operations should never be disconnected from meaning. Division of fractions, for example, is still a form of division and should connect to a meaning of division.

- **<u>Place Value:</u>** Knowledge of place value is essential, and students should develop ways of thinking about place value that enable them to see the relationships between places. For example, they can think of a value in one place as 10 times that same value in the place to the right (or a *bundle of 10*), and they can carry that thinking between any places in any direction. They should be able to use that understanding effortlessly to compose and decompose quantities and to connect place-value understanding to operations.

- **Comparisons:** Comparisons can be additive or multiplicative, with context guiding which is most appropriate. A multiplicative comparison is relative, describing one quantity in terms of another (e.g., 6 meters is 3 times as large as 2 meters). Additive comparisons are absolute; the comparison is based on some other quantity (e.g., 6 meters is 4 meters more than 2 meters). Students should have ways of thinking that help them determine which comparison to use or how an existing comparison is additive or multiplicative.

- **Measurement:** Geometric measurement is ultimately understood as the result of a multiplicative comparison between common attributes of two measurable quantities, and the result describes how many copies of a are contained in b. Equivalently, measurement addresses a times-as-large comparison such as "a is n times as large as b." Students thinking about measurement should have a clear understanding of which attribute is being measured and the comparison of two objects with that attribute, where one object's attribute is measured in terms of the other.

- **Fractions:** A fraction is a single number. It is a number just as 1, or 100, or 37,549 are, and it has a location on the number line. Students should be able to think of a fraction as a number and treat it as such. The fraction a/b can be thought of as a copies of $1/b$, where $1/b$ is the length of a single part when the interval from 0 to 1 is partitioned into b parts. Two fractions are equivalent when they share a location on the number line.

- **Formulas:** Mastery of formulas (and procedures) is not the goal. Formulas are not the mathematics; they should be seen as shortcuts to help accomplish something with the mathematics. Students should have ways of thinking about the formulas that enable them to make sense of the quantities and to determine why quantities are connected with the indicated operations. Students should have mental and mathematical ways to reinvent useful formulas (e.g., $A = \pi r^2$ means three-and-a-bit copies of the square with area r^2).

- **Variables:** Students should have ways of thinking that enable them to distinguish between unknowns that vary and unknowns that represent some fixed value. For example, in the equation $6 = 3x + 2$, x represents some unknown fixed value that makes the equation true. In $y = 3x + 2$, y and x vary with each other. In $y = mx + b$, y and x vary with each other, while m and b are typically nonvarying constants (parameters) within a problem.

- **Covarying Quantities:** Single quantities can vary, but students also need to consider situations where two quantities vary together. For example, in the equation $y = 3x + 2$, as the quantity x varies, the quantity y varies. It can be helpful to think about the relationship between covarying quantities in terms of how changes in x result in changes in y (e.g., as x increases by 1, y changes by . . .).

- **Proportional Relationships:** Proportional relationships require two covarying quantities. Those quantities must be measurable in some way, and the measures of those quantities scale in tandem. When one quantity changes by a scale factor, the other quantity also changes by the same scale factor. For example, doubling one quantity's value results in a doubling of the other quantity's value. Students should have ways of thinking that allow them to distinguish the two varying quantities in any proportional relationship and to explain how the quantities change by the same scale factor. It is important to note that proportional relationships are not synonymous with proportions. See this for more information.

- **The Equal Sign:** The equal sign works in multiple ways in mathematics, even though the symbol does not change. Students should be able to think about the symbol as being relational (e.g., $2 + 4 = 5 + 1$) and as being operational (e.g., the output of a computation), and should be able to determine which role the symbol is playing based on the situation. Keep an eye out for when students may put together equal signs. If this happens, prompting the student with questions about what the equals sign means may help. For example, suppose a student writes: 9 x 8 = 72 = 126 – 72 = 54 (*in this case, the student might be using the equal sign to say "next, I will..."*).

How to Conduct a Formative Conversation[1]

1. The questions were developed to help teachers elicit information about students' ways of thinking about the content in the item and about mathematical ideas. These questions are suggestions, however, and not intended to be used as a script. The conversations teachers have will vary by student. While the questions for an item are laid out in a progression, teachers can vary the order to adapt to students' responses. Teachers should also keep in mind that students' responses may point to ways of thinking that are not addressed by the provided questions. In these cases, teachers might pursue those student understandings with their own line of questioning. There are several actions teachers can take to prepare for formative conversations:
    a. Become familiar with the questions ahead of time. This ensures that teachers can select the most appropriate next question based on how the students are responding.
    b. Provide students with tools to help them answer the questions. Depending on the task, these tools might include a manipulative, drawing paper, graph paper, or individual whiteboards and markers.
    c. Have a list of questions to help further probe what students are thinking. Some examples are:
        i. Can you tell me more about that?
        ii. You look like you're really thinking about this. What are you thinking?
        iii. Can you draw me a picture/write an equation?
        iv. How did you get that answer?
        v. Is there another way that you could find that answer?
    d. Make a plan to track what students say during the conversation: record the conversation, take notes, or have an observer take notes.

[1] The final, freely available Formative Conversation Starters collection will provide grade-level, standard-aligned MAP Growth items to serve as an entry point, and then call out mathematical ideas and the ways of thinking associated around those items. Each item will include multiple clusters to address multiple BINS. Some clusters target core understandings that underpin the content. Other clusters elicit flexibility of thinking or extend beyond the assessment item being discussed.

2. Teachers should ask questions without judgment – the goal is to listen. Student responses should not be labeled as right or wrong, as this is not the time for that, and follow-up questions should be asked regardless of whether students give a correct response. Teachers should avoid commenting on students' responses other than to ask follow-up questions or to clarify what a student has said. Other students, however, might be encouraged to agree or disagree in a small group setting.
3. One of the most important parts of the formative conversation is what comes after the conversation: how will a teacher use the information about student thinking when planning instruction. Consider these suggestions for how to act on a formative conversation:
    a. Identify the different mathematical ideas addressed in the formative conversation. Where did students make connections between the ideas? Where do the connections need to be strengthened?
    b. Identify what students already understand in order to build instruction on that understanding.
    c. Identify areas where students can deepen what they already understand.

Sample Formative Conversation Starters

Each section below contains a sample group of questions that connect strongly to one of the BINS. The appendix includes the same questions along with annotations that include suggestions of things to listen for, clarifications on the purposes of the questions, and potential student responses. This is only a sample collection from the set we plan to release in Fall, 2021.

1. Operations

- In mathematics, what does division do for us?
    - Without computing, what is one meaning of division you can use to make sense of $10 \div (0.2)$ (or $10 \div \frac{1}{5}$)?
    - Without computing, what is one meaning of division you can use to make sense of $(-10) \div (0.2)$ (or $(-10) \div \frac{1}{5}$)?
    - Without computing, describe how you can estimate the value of $(-10) \div (-0.3)$ using division as comparison (copies of, or times as large language).

2. Place Value

- How does place value work?
    - Someone tells you that 2,001 = 21 because zeros have no value. What do you say?
    - How does 200 compare to 2,000? Which comparison connects to place value?
    - What is the relationship between the 2s in each number below? Can you describe the relationship in two ways?

5,221	2,521

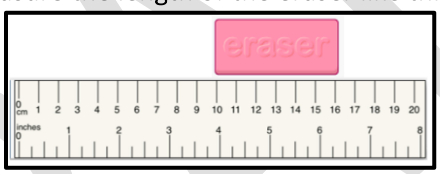
3. Comparisons

- How many different ways can you compare two numbers?
    - Compare 12 to 3 in as many ways as you can.
    - Complete these statements:
    - 12 is _____ times as large as 3.
    - 3 is _____ times as large as 12.
    - Compare 12 to 8 in as many ways as you can.
    - Compare 11 to 3 in as many ways as you can.
    - Tell me when you see a comparison in each of the following equations. What is being compared?

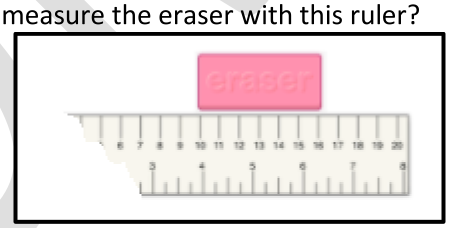
$$13 - \square = 8$$
$$15 - 8 = \square$$
$$12 \times 42 = \square$$
$$14 \div \square = 3$$
$$14 \div 5 = \square$$

4. Measurement

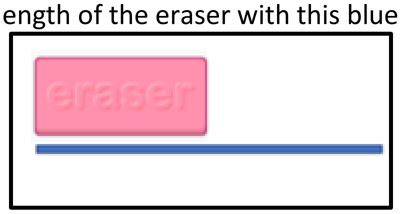
- Suppose someone tries to measure the length of the eraser like this. What would you tell them?



- Tell me when you see that the eraser in the diagram is as long as six copies of one centimeter.
- Tell me when you can see that the eraser is about 2 copies of one inch.
- If a student is firm on the need to start at zero, hand them a paper ruler with the left end cut and ask: Can you measure the eraser with this ruler?



- Can you explain what it means to measure the length of this eraser?
- Could we measure the length of the eraser with this blue line?



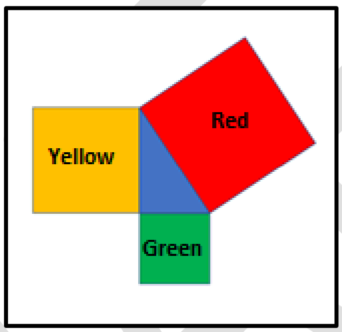
- What might be the measure in “blue lines”?

5. Fractions

- What is a fraction?
  - Is a fraction two numbers or one number?
  - What other words are used when talking about fractions? What do they mean?
  - What do you think of when you see the fraction $\frac{3}{8}$? (If it means "3 out of 8," then what does 8/3 mean?)
  - Can you write 3/8 as a sum of fractions?
  - Can you write 3/8 as a product of fractions?

6. Formulas

- What does this figure have to do with the Pythagorean theorem?



  - What happens if $c^2 > a^2 + b^2$? How can this diagram help you explain it?
  - What happens if $c^2 < a^2 + b^2$? How can this diagram help you explain it?

7. Variables

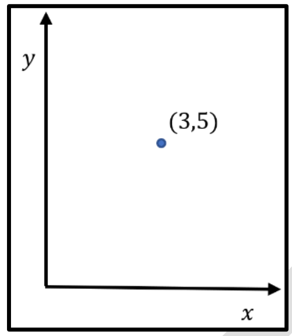
- How are these equations alike? How are they different?

$$3x + 5(2) = 15 \qquad\qquad 3x + 5y = 15$$

  - How would you "solve" the first equation?
  - How would you "solve" the second equation?

8. Covarying Quantities

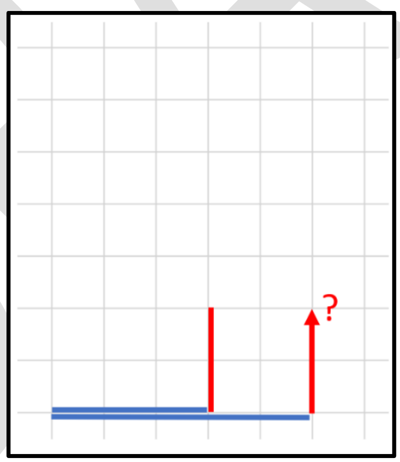
Look at this graph.



- If this point is on a line with a slope of –2, what is y when x is 2? Explain.
  - What is y when x is 0? Explain.
  - What is an equation for this line?
  - How can you find the equation of any line when you have the slope and any point?

9. Proportional Relationships

- In the image below, the length of the blue line increased as shown. If the length of the red line is proportional to the length of the blue line, what is the length of the longer red line? Can you explain it without using a formula?



  - Are there any meaningful multiplicative comparisons in the above diagram?
  - What is meaningful about the number $\frac{5}{3}$ in the diagram above?
  - What is meaningful about the number $\frac{2}{3}$ in the diagram above?
  - If we imagine the blue line growing across all possible lengths, what happens with the red line? Are you sure?
  - Can you write an equation that shows the relationship between the lengths of the blue and red lines?

10. The Equal Sign

Let's look at some equations with unknowns:

$2 + 9 = \square$

$5 + \square = 8$

$2 + 6 = \square + 3$

- What does the box, or unknown, mean in each equation?
    - What does the equal sign mean in each equation? Are the three equal signs different in any way?

Appendix: Sample Formative Conversation Starters (Annotated)

1. Operations

In mathematics, what does division do for us?

Division tells size of groups, or number of groups. It also provides multiplicative comparisons (telling times as large as).

Without computing, what is one meaning of division you can use to make sense of $10 \div (0.2)$ (or $10 \div \frac{1}{5}$)?

Possible solutions: How many 1/5's are in 10, or 10 is how many times as large as 1/5, or 10 is 1/5 of what whole group.

Without computing, what is one meaning of division you can use to make sense of $(-10) \div (0.2)$ (or $(-10) \div \frac{1}{5}$)?

Possible solutions: How many 1/5's are in –10, or –10 is how many times as large as 1/5, or – 10 is 1/5 of what whole group.

Without computing, describe how you can estimate the value of $(-10) \div (-0.3)$ using division as comparison (copies of, or times as large language).

The expression tells how many copies of (-0.3) there are in (-10). Given that 0.3 is close to 1/3, a ballpark estimate would be $10 \div \frac{1}{3} = 30$.

2. Place Value

How does place value work?

A value in one place is 10 times that same value in the place to its right (or a bundle).

Someone tells you that 2,001 = 21 because zeros have no value. What do you say?

The value of a digit in a number is based on its place in the number. In both 2,001 and 21, there is 1 one. The zeros in 2,001 make it look like there are no tens and no hundreds, but actually there are 200 tens or 20 hundreds. The number 21 has 2 tens.

How does 200 compare to 2,000? Which comparison connects to place value?

2,000 is 1,800 more than 200. It is also 10 times as great as 200.

What is the relationship between the 2s in each number below? Can you describe the relationship in two ways?

5,221	2,521

5,221: The 2 in the hundreds place is 10 times as great as the 2 in the tens place. The 2 in the tens place is one-tenth as great as the 2 in the hundreds place.

2,521: The 2 in the thousands place is 100 times as great as the 2 in the tens place. The 2 in the tens place is one-hundredth as great as the 2 in the thousands place.

3. Comparisons

How many different ways can you compare two numbers?

Additive or multiplicative comparison can be used. Also, greater than or less than can be used to compare numbers.

Compare 12 to 3 in as many ways as you can.

Additive comparison: 12 is 9 more than 3. Multiplicative comparison: 12 is 4 times as much as 3. Also, “greater than” language can be used: 12 is greater than 3.

Complete these statements:

- 12 is _____ times as large as 3.
  4
- 3 is _____ times as large as 12.
  1/4

Compare 12 to 8 in as many ways as you can.

Additive comparison: 12 is 4 more than 8. Multiplicative comparison: 12 is 1.5 times as much as 3. Also, “greater than” language can be used: 12 is greater than 3.

Compare 11 to 3 in as many ways as you can.

Additive comparison: 11 is 9 more than 3. Multiplicative comparison, 11 is 3 2/3 times as many as 3.

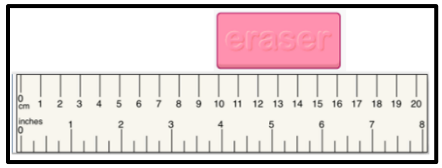
Look at these equations. Tell me when you see a comparison in the following equations. What is being compared?

$$13 - \square = 8$$
$$15 - 8 = \square$$
$$12 \times 42 = \square$$
$$14 \div \square = 3$$
$$14 \div 5 = \square$$

For the first, one might say 13 is 8 more than the number in the box – an additive comparison. For the second, the unknown in the box represents how much greater 15 is than 8 (also additive). For the third the unknown in the box is 12 times as large as 42 (multiplicative). In the fourth 14 is 3 times the size of unknown in the box (multiplicative). In the fifth the box tells me how many copies of 5 are in 14 (multiplicative). It may be necessary to nudge students for a response that tells how the box works in the comparison.

4. Measurement

Suppose someone tries to measure the length of the eraser like this. What would you tell them?



Listen for students' understanding of measurement. Some students may say "move the eraser to start at zero." Some students may say it's ok to measure this way.

Tell me when you see that the eraser in the diagram is as long as six copies of one centimeter.

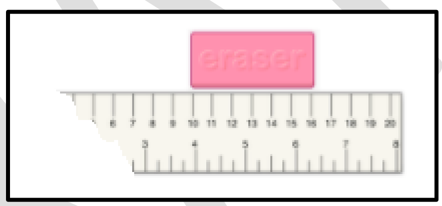
A student may see that there are 6 lengths of one centimeter in the length of the eraser. Or a student may see 16 – 10 = 6.

Tell me when you can see that the eraser is about 2 copies of one inch.

A student may say that there are 2 lengths of one inch in the length of the eraser. A student may see 6 – 4 = 2. It is also common for students to respond with, "first I would need to move the eraser to start at zero."

If a student is firm on the need to start at zero, hand them a paper ruler with the left end cut and ask:

Can you measure the eraser with this ruler?

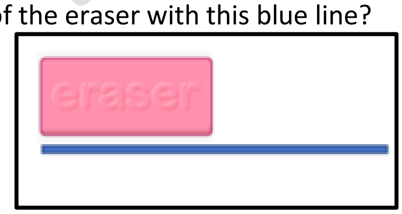


Yes, it has a length.

Can you explain what it means to measure the length of this eraser?

We are comparing the length of the eraser to the length of 1 centimeter or 1 inch to see how many copies of 1 centimeter or 1 inch long the length of the eraser is.

Could we measure the length of the eraser with this blue line?



Yes, it has a length.

What might be the measure in "blue lines"?

It's drawn to be one-half of a blue line, so the length of the eraser might be one-half of a blue line.

5. Fractions

What is a fraction?

A fraction is a number, just as 3, 12, or 0 are numbers. It represents a point on the number line.

Is a fraction two numbers or one number?

A fraction looks like two numbers, but it is a single number.

What other words are used when talking about fractions? What do they mean?

Numerator and denominator, for example. The denominator tells the number of partitions of the whole, the numerator tells how many copies of that partition we have.

What do you think of when you see the fraction $\frac{3}{8}$?

Do your students first think of fractions as numbers on a number line? Do they see them as pieces of a pie? As parts of a set? As fraction bars?

(If it means "3 out of 8" to a student, then what would 8/3 mean?)

Can you write 3/8 as a sum of fractions?

Yes. For example, 1/8 + 1/8 + 1/8.

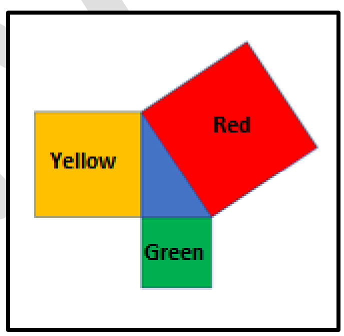
Can you write 3/8 as a product of fractions?

Fractions may be viewed additively or multiplicatively. Ideally, students have agility representing fractions in both ways. For example, 3/8 can be written as 2/8 + 1/8, and students may also see 3/8 as 3 copies of 1/8, or 3 x 1/8.

6. Formulas

What does this figure have to do with the Pythagorean theorem?

The sum of the squares of the lengths of the two shorter sides is equal to the square of the length of the third side. That is, the area of the yellow square plus the area of the green square is equal to the area of the red square.



What happens if $c^2 > a^2 + b^2$? How can this diagram help you explain it?

The triangle would have one obtuse angle. The diagram might help by showing that since the area of the red square is greater than the sum of the areas of the yellow and green squares, the angle across from c is greater than 90 degrees.

What happens if $c^2 < a^2 + b^2$? How can this diagram help you explain it?

The triangle would have three acute angles. The diagram might help explain by showing that since the area of the red square is less than the sum of the areas of the yellow and green squares, angle across from c is less than 90 degrees.

7. Variables

How are these equations alike? How are they different?

They are alike in that both equations show that the expressions on the left side of the equations are both equal to 15 and both contain the term, $3x$. The equation on the left is the special case for the one on the right when $y = 2$. They are different in that in the first equation there is one unknown quantity, x. In the second equation there are two unknown quantities, x and y. In the single variable equation, there is a single value of x that makes the equation true. In the two-variable equation, there are many values for x and y that make the equation true.

$3x + 5(2) = 15$	$3x + 5y = 15$

How would you "solve" the first equation?

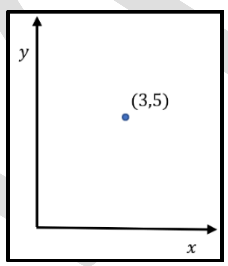
One possibility: Think $5 \times 2 = 10$. Then think, $5 + 10 = 15$. Therefore, since $3x + 10 = 15$, it follows that $3x = 5$. This leads to the solution, $x = \frac{5}{3}$.

How would you "solve" the second equation?

We would need to find all pairs of values that make the equation true. Find all values of y such that $y = \frac{(15-x)}{5}$ or $y = 3 - \left(\frac{3}{5}\right)x$. That is, all ordered pairs $(x, 3 - \frac{3}{5}x)$ show solutions to the equation. A graph is an ideal way to represent all of those solutions.

8. Covarying Quantities

Look at this graph.



If this point is on a line with a slope of –2, what is y when x is 2? Explain.

With a slope of –2, if x increases 1 unit, then y will decrease by 2 units. Or, if x decreases by 1 unit, y will increase by 2 units. Therefore, the line passing through (3, 5) will also pass through (2, 7). Note that we decreased x by 1 unit, which increased y by 2 units (from $y = 5$ to $y = 7$).

What is y when x is 0? Explain.

When $x = 0$, $y = 11$. With a slope of –2, decreasing x by 2 units from $x = 2$ to $x = 0$ will create an increase in y of 4 units from $y = 7$ to $y = 11$.

What is an equation for this line?

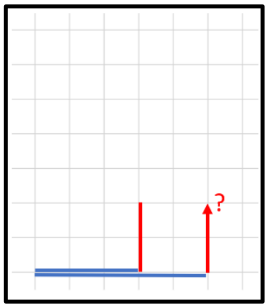
$y = 11 - 2x$

How can you find the equation of any line when you have the slope and any point?

Are students able to think this through based on meanings rather than formulas? One strategy is to find the vertical intercept by leveraging the slope meaning and the known point. Then, we can express the equation of the line with slope a and vertical intercept $(0, b)$ as $y = ax + b$.

9. Proportional Relationships

In the image below, the length of the blue line increased as shown. If the length of the red line is proportional to the length of the blue line, what is the length of the longer red line? Can you explain it without using a formula?



The length of the new blue line is $\frac{5}{3}$ the length of the original blue line. So, the length of the new red line must also be $\frac{5}{3}$ the length of the original red line, which is of 2, or $\frac{10}{3}$.

Are there any meaningful multiplicative comparisons in the above diagram?

We can compare, multiplicatively, the length of the original blue line (3 units) and the increased length of the blue line (5 units). The increased length (5 units) is $\frac{5}{3}$ (or $1\frac{2}{3}$) times the length of the original length (3 units). Equivalently, the original length (3 units) is $\frac{3}{5}$ times the increased length (5 units). Alternatively, we can compare the length of the original blue line (3 units) with the length of the original red line (2 units) by claiming that the red lines' length is $\frac{2}{3}$ times the length of the blue line. Reciprocally, the blue line's length (3 units) is $\frac{3}{2}$ times the red line's length (2 units).

What is meaningful about the number $\frac{5}{3}$ in the diagram above?

It represents the ratio of the length of the longer blue line (5 units) to the length of the original blue line (3 units). It can be thought of as the scale factor between the lengths. It represents the multiplicative comparison of the increased length of the blue line (5 units) and the original length of the blue line (3 units).

What is meaningful about the number $\frac{2}{3}$ in the diagram above?

It represents the ratio of the length of the original red line (2 units) to the length of the original blue line (3 units). It can be thought of as the scale factor between the lengths. It represents the multiplicative comparison of the original length of the red line (2 units) and the original length of the blue line (3 units).

If we imagine the blue line growing across all possible lengths, what happens with the red line? Are you sure?

Like the blue line, the red line will grow in a way that keeps the ratio between the lengths constant. If the blue line doubles in length, so will the red line. If the blue line increases its length by a factor of 1.1 times, so will the red line. Likewise, the length of the red line will always be $\frac{2}{3}$ of the length of the blue line.

Can you write an equation that shows the relationship between the lengths of the blue and red lines?

If b is the length of the blue line and r is the length of the red line, then $r = \frac{2}{3} \cdot b$. Or, $b = \frac{3}{2} \cdot r$.

10. The Equal Sign

Let's look at some equations with unknowns:

2 + 9 = □
5 + □ = 8
2 + 6 = □ + 3

What does the box, or unknown, mean in each equation?

It represents a value we don't know. In the first equation, the box is equal in value to the sum of 2 and 9. In the second equation, the box is the value that when increased by 5, equals 8. In the last equation, the box represents the value that when increased by 3, is equal to 2 + 6.)

What does the equal sign mean in each equation? Are the three equal signs different in any way?

In each of the equations, the equal sign is a statement that one side is equal in value to the other side. Be sure that students can flexibly interpret the equal sign as relational. A purely operational perspective, such as "the result when you add… ," will lead to difficulty when looking at the third equation, for example.